

# HARNACK INEQUALITIES FOR ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY LÉVY PROCESSES

**ABSTRACT.** By using the existing sharp estimates of density function for rotationally invariant symmetric  $\alpha$ -stable Lévy processes and rotationally invariant symmetric truncated  $\alpha$ -stable Lévy processes, we obtain that Harnack inequalities hold for rotationally invariant symmetric  $\alpha$ -stable Lévy processes with  $\alpha \in (0, 2)$  and Ornstein-Uhlenbeck processes driven by rotationally invariant symmetric  $\alpha$ -stable Lévy process, while logarithmic Harnack inequalities are satisfied for rotationally invariant symmetric truncated  $\alpha$ -stable Lévy processes.

**Keywords:** Harnack inequalities; logarithmic Harnack inequalities; Ornstein-Uhlenbeck processes;  $\alpha$ -stable Lévy processes

**MSC 2010:** 60J25; 60J51; 60J52.

## 1. MAIN RESULTS

The dimension free Harnack inequality for diffusion semigroups was first established in (Wang, 1997) under a curvature condition. Using the coupling method and Girsanov transformations, this inequality has been derived in (Arnaudon et al., 2006) for diffusions with curvature unbounded below, also see e.g. (Wang, 2007, 2010b; Ouyang, 2009) and references therein for recent work on this topic. These methods are applied in (Röckner and Wang, 2003; Ouyang et al., 2011) to study Harnack inequalities for a class of Ornstein-Uhlenbeck type processes driven by Lévy process with non-degenerate Gaussian noise on Hilbert spaces. In this paper we will establish explicit Harnack inequalities for Ornstein-Uhlenbeck processes driven by pure Lévy jump processes. For this aim, we will make full use of the existing estimates of density functions for Lévy processes.

Let  $L_t$  be a  $d$ -dimensional Lévy process starting from the origin and  $A$  be a  $d \times d$  matrix. The corresponding *Ornstein-Uhlenbeck process* is defined as the unique strong solution of the stochastic differential equation

$$dX_t = AX_t dt + dL_t.$$

Denote by  $P_t$  the semigroup of  $X_t$ , and by  $B_b^+(\mathbb{R}^d)$  the set of all bounded and nonnegative measurable functions on  $\mathbb{R}^d$ . Our main contribution is as follows:

**Theorem 1.1.** *Assume that the Lévy measure of  $L_t$  satisfies*

$$\nu(dz) \geq c|z|^{-d-\alpha} dz,$$

---

*J. Wang:* School of Mathematics and Computer Science, Fujian Normal University, 350007, Fuzhou, P.R. China and TU Dresden, Institut für Mathematische Stochastik, 01062 Dresden, Germany. jianwang@fjnu.edu.cn.

where  $\alpha \in (0, 2)$  and  $c > 0$ . Then there exists  $C > 0$  (only depending on  $d, \alpha, c$  and  $\|A\|$ ) such that for any  $t > 0, x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,

$$(1.1) \quad P_t f(x) \leq C P_t f(y) \left( 1 + \frac{|x - y|}{(t \wedge 1)^{1/\alpha}} \right)^{d+\alpha}.$$

If moreover  $A = 0$ , then

$$P_t f(x) \leq C P_t f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}.$$

**Remark 1.2.** We shall point out that the above type of Harnack inequality does not hold for Brownian motion. That is, let  $P_t$  be the semigroup of Brownian motion on  $\mathbb{R}^d$ . Then, there does not exist a finite measurable function  $\Phi$  on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  such that for any  $t > 0, x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,

$$P_t f(x) \leq P_t f(y) \Phi(t, x, y).$$

Indeed, such inequality is not true for most diffusion semigroups on noncompact manifolds, e.g. see (Wang, 2006a).

By the Hölder inequality, Theorem 1.1 immediately yields the following result, which indicates that *F.-Y. Wang's Harnack inequalities* hold for rotationally invariant symmetric  $\alpha$ -stable Lévy processes with  $\alpha \in (0, 2)$  and Ornstein-Uhlenbeck processes driven by rotationally invariant symmetric  $\alpha$ -stable Lévy process.

**Corollary 1.3.** *Under the condition in Theorem 1.1, there exists a constant  $C > 0$  such that for any  $t > 0, p > 1, x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,*

$$(1.2) \quad \left( P_t f(x) \right)^p \leq C P_t f^p(y) \left( 1 + \frac{|x - y|}{(t \wedge 1)^{1/\alpha}} \right)^{p(d+\alpha)}.$$

If moreover  $A = 0$ , then

$$\left( P_t f(x) \right)^p \leq C P_t f^p(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{p(d+\alpha)}.$$

Comparing with (Gordina et al., 2011, Theorem 1.1) where a dimension free Harnack inequality is presented for  $\alpha \in (1, 2)$ , the advantage of our results is that they hold for any  $p \geq 1$  and  $\alpha \in (0, 2)$ . On the other hand, however, our results are dimension-dependent so that they can not be extended to infinite dimensions. Next, (Gordina et al., 2011, Theorem 1.4) includes a dimension-free log-Harnack inequality for  $\alpha \in (0, 1]$ . Below we present a simpler but dimension-dependent version of this inequality for a truncated  $\alpha$ -stable Lévy process. We refer to (Bobkov et al., 2001; Wang, 2010a) for the background and some properties of logarithmic Harnack inequalities.

**Theorem 1.4.** *Assume that the Lévy measure of  $L_t$  satisfies*

$$\nu(dz) \geq c|z|^{-d-\alpha} \mathbf{1}_{\{|z| \leq r\}} dz,$$

where  $\alpha \in (0, 2)$  and  $c, r \in (0, \infty)$ . Let  $P_t$  be the semigroup of  $L_t$ . Then there exists  $C > 0$  (only depending on  $d, \alpha, c$  and  $r$ ) such that for any  $t > 0, x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$ ,

$$P_t(\log f)(x) \leq \log P_t f(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right).$$

**Note** The classical Harnack's inequality is an inequality relating the values of a positive harmonic function at two points. In this paper, we also call (1.1) or (1.2) the Harnack inequality. The notation, different from the classical one, is due to Prof. F.-Y. Wang's work. The readers can refer to the survey paper (Wang, 2006b) for more details of Wang's Harnack inequality.

## 2. PROOFS

The idea of the proofs of Theorems 1.1 and 1.4 is to compare the original process with rotationally invariant symmetric  $\alpha$ -stable Lévy processes and with rotationally invariant symmetric truncated  $\alpha$ -stable Lévy processes, respectively.

**2.1. Proof of Theorem 1.1.** We first present some preliminary results concerning rotationally invariant symmetric  $\alpha$ -stable Lévy process  $X_t$ , which is a pure jump Lévy process on  $\mathbb{R}^d$  with Lévy measure

$$(2.3) \quad \nu(dz) = c|z|^{-(d+\alpha)} dz,$$

where  $c > 0$  and  $\alpha \in (0, 2)$ .

According to (Blumenthal and Gettoor, 1960, Theorem 2.1), we know that the process  $X_t$  has a continuous density function  $p_t(x, y)$ , and there exist constants  $c_1$  and  $c_2 > 0$  that depend only on  $d, \alpha$  and the constant  $c$  in (2.3) such that for all  $x, y \in \mathbb{R}^d$  and  $t > 0$ ,

$$(2.4) \quad c_1 \min \left\{ t^{-d/\alpha}, \frac{t}{|x - y|^{d+\alpha}} \right\} \leq p_t(x, y) \leq c_2 \min \left\{ t^{-d/\alpha}, \frac{t}{|x - y|^{d+\alpha}} \right\}.$$

The following result is based on (2.4), and it is a key in the proof of Theorem 1.1.

**Lemma 2.1.** *Let  $p_t(x, y)$  be the density function of the rotationally invariant symmetric  $\alpha$ -stable Lévy process  $X_t$  above. Then, for any  $t > 0$  and  $x, y, z \in \mathbb{R}^d$ ,*

$$(2.5) \quad \frac{p_t(x, z)}{p_t(y, z)} \leq \frac{2^{\alpha+d} c_2}{c_1} \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha},$$

*Proof.* (i) When  $|y - z| \leq t^{1/\alpha}$ ,

$$t^{-d/\alpha} \leq \frac{t}{|y - z|^{\alpha+d}},$$

and so

$$p_t(y, z) \geq c_1 t^{-d/\alpha}.$$

Then,

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_2 t^{-d/\alpha}}{c_1 t^{-d/\alpha}} = \frac{c_2}{c_1}.$$

(ii) If  $|y - z| \geq 2(t^{1/\alpha} \vee |x - y|)$ , then

$$|x - z| \geq |y - z| - |x - y| \geq |y - z| - |y - z|/2 = |y - z|/2 \geq t^{1/\alpha}.$$

We get

$$t^{-d/\alpha} \wedge \frac{t}{|x - z|^{\alpha+d}} = \frac{t}{|x - z|^{\alpha+d}} \leq \frac{2^{\alpha+d}t}{|y - z|^{\alpha+d}}.$$

On the other hand,

$$t^{-d/\alpha} \wedge \frac{t}{|y - z|^{\alpha+d}} = \frac{t}{|y - z|^{\alpha+d}}.$$

Therefore,

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_2 2^{\alpha+d}}{c_1}.$$

(iii) If  $|y - z| \leq 2(t^{1/\alpha} \vee |x - y|)$  and  $|y - z| > t^{1/\alpha}$ , then

$$t^{-d/\alpha} \wedge \frac{t}{|y - z|^{\alpha+d}} = \frac{t}{|y - z|^{\alpha+d}}.$$

So,

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_2 t^{-d/\alpha}}{c_1 t / |y - z|^{\alpha+d}} = \frac{c_2 |y - z|^{\alpha+d}}{c_1 t^{(d+\alpha)/\alpha}}.$$

Noticing that in this case

$$\frac{|y - z|}{t^{1/\alpha}} \leq 2 \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right),$$

we have

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{2^{\alpha+d} c_2}{c_1} \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}.$$

Combining with all the estimates above, we can arrive at (2.5). The proof is completed.  $\square$

Having Lemma 2.1 at hand, we can easily obtain the following statement for rotationally invariant symmetric  $\alpha$ -stable Lévy processes.

**Proposition 2.2.** *Let  $P_t$  be the semigroup of the rotationally invariant symmetric  $\alpha$ -stable Lévy process  $X_t$  above. Then there exists  $C > 0$  such that for any  $t > 0$ ,  $x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,*

$$(2.6) \quad P_t f(x) \leq C P_t f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}.$$

*Proof.* Let  $P(t, x, dz)$  be the transition probability function of  $X_t$ . We find that for any  $f \in B_b^+(\mathbb{R}^d)$  and  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} P_t f(x) &= \int f(z) P(t, x, dz) = \int f(z) p_t(x, z) dz \\ &= \int f(z) \frac{p_t(x, z)}{p_t(y, z)} p_t(y, z) dz \\ &\leq \int f(z) p_t(y, z) dz \max_{z \in \mathbb{R}^d} \frac{p_t(x, z)}{p_t(y, z)} \\ &\leq C P_t f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}. \end{aligned}$$

The proof is completed.  $\square$

**Remark 2.3.** According to the proof of Proposition 2.2, the inequality (2.6) is an consequence of (2.4). Thus, (2.6) is also satisfied for symmetric (not necessarily rotationally invariant)  $\alpha$ -stable Lévy processes, see e.g. (Bogdan et al., 2003).

Next, we turn to the proof of Theorem 1.1.

*Proof of Theorem 1.1.* (1) We first assume that  $A = 0$ . Since  $\nu(dz) \geq c|z|^{-d-\alpha}dz$ , we can write

$$X_t = X'_t + X_t^S,$$

where  $X_t^S$  is a rotationally invariant symmetric  $\alpha$ -stable Lévy process with Lévy measure

$$\nu_{X^S}(dz) := c|z|^{-d-\alpha}dz,$$

and  $X'_t$  is a Lévy process with Lévy measure

$$\nu_{X'}(dz) := \nu(dz) - c|z|^{-d-\alpha}dz \geq 0.$$

Let  $P_t$ ,  $P_t^S$  and  $P'_t$  be the semigroups of the Lévy processes  $X_t$ ,  $X_t^S$  and  $X'_t$ , respectively. Then, according to Proposition 2.2, there exists  $C > 0$  such that for any  $t > 0$ ,  $x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,

$$P_t^S f(x) \leq C P_t^S f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}.$$

Therefore, for  $t > 0$ ,  $x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$ ,

$$\begin{aligned} P_t f(x) &= P_t^S P'_t f(x) \\ &\leq C P_t^S P'_t f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha} \\ &= C P_t f(y) \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}, \end{aligned}$$

where in the above inequality we have used the fact that  $P'_t f \in B_b^+(\mathbb{R}^d)$ . The desired assertion (1.1) follows.

(2) Suppose that  $A \neq 0$ . We borrow an idea from the proof of (Wang, 2011, Theorem 1.1). It is well known that for the semigroup of Ornstein-Uhlenbeck process

$$P_t f(x) = \int f(e^{tA}x + y) \mu_t(dy),$$

where  $\mu_t$  is the probability measure on  $\mathbb{R}^d$  with characteristic function

$$\hat{\mu}_t(\xi) = \exp \left[ \int_0^t \Phi(e^{sA^*} \xi) ds \right],$$

and  $\Phi$  is the symbol of the Lévy process  $L_t$ , cf. see (Jacob, 2001). Define

$$c_0 = \int_{\{|z| \leq e^{-\|A\|}\}} (1 - \cos z_1) |z|^{-d} dz.$$

Then, following Part (III) in the proof of (Wang, 2011, Theorem 1.1), for any  $s \in (0, 1]$ ,

$$\xi \mapsto \Phi(e^{sA^*} \xi) ds - c_0 c |\xi|^\alpha$$

is a Lévy symbol. Thus, for every  $t \in (0, 1]$  there exists a probability measure  $\pi_t$  on  $\mathbb{R}^d$  with

$$\hat{\pi}_t(\xi) = \exp \left[ \int_0^t \Phi(e^{sA^*} \xi) ds - t c_0 c |\xi|^\alpha \right].$$

Let  $P_t^S$  be the semigroup for the Lévy process with symbol  $c_0 c |\xi|^\alpha$ , and define

$$P'_t f(x) = \int f(x + z) \pi_t(dz).$$

We have

$$P_t f(x) = P_t^S P'_t f(e^{tA}x).$$

Note that  $P_t^S$  is also the semigroup of a rotationally invariant symmetric  $\alpha$ -stable Lévy process with Lévy measure

$$\nu_S(dz) := c_1 |z|^{-d-\alpha} dz.$$

Then the required assertion for  $t \in (0, 1]$  follows from the arguments in step (1).

For any  $t > 0$  and  $f \in B_b^+(\mathbb{R}^d)$ ,

$$\begin{aligned} P_t f(x) &= P_{t \wedge 1} P_{(t-1)^+} f(x) \\ &\leq C P_{t \wedge 1} P_{(t-1)^+} f(y) \left( 1 + \frac{|x - y|}{(t \wedge 1)^{1/\alpha}} \right)^{d+\alpha} \\ &= P_t f(y) \left( 1 + \frac{|x - y|}{(t \wedge 1)^{1/\alpha}} \right)^{d+\alpha}, \end{aligned}$$

where in the inequality above we have used again the fact that  $P_{(t-1)^+} f \in B_b^+(\mathbb{R}^d)$ . The proof is end.  $\square$

To end up this section, we further present two remarks about Theorem 1.1.

**Remark 2.4.** (1) The proof of Theorem 1.1 yields that (1.1) and (1.2) hold for the following two classes of Lévy type processes: (i) *Stable-like processes on  $\mathbb{R}^d$* , whose Lévy jump kernel is given by  $j(x, y) = \frac{c(x, y)}{|x - y|^{d+\alpha}}$  and  $c(x, y)$  is a symmetric function on  $\mathbb{R}^d \times \mathbb{R}^d$  that is bounded upper and below by two positive constants. More details about stable-like processes are referred to (Chen and Kumagai, 2003, Theorem 1.1) and (Chen and Kumagai, 2008, Theorem 1.2). (ii) *Markov processes on  $\mathbb{R}^d$  generated by  $\Delta^{\alpha/2} + b(x) \cdot \nabla$* , where  $1 < \alpha < 2$  and  $b$  in the Kato class  $\mathcal{K}_d^{\alpha-1}$  on  $\mathbb{R}^d$ . The readers can see (Bogdan and Jakubowski, 2007, Theorem 2) for the comparability between the transition density for the small time of these processes and that of rotationally invariant symmetric  $\alpha$ -stable Lévy processes.

(2) As shown in (Wang, 1997, 2006b, 2007, 2010a), to derive contractivity properties of the semigroup from the Harnack inequality, one needs concentration properties of the associated invariant measure, which is however unknown for the present setting.

**2.2. Proof of Theorem 1.4.** We begin with recalling some results about a rotationally invariant symmetric truncated  $\alpha$ -stable Lévy process  $X_t$ . The corresponding Lévy measure is given by

$$\nu(dz) = c|z|^{-(d+\alpha)} \mathbb{1}_{\{|z| \leq r\}} dz,$$

where  $c, r \in (0, \infty)$  and  $\alpha \in (0, 2)$ . Let  $p_t(x, y)$  be the density function of  $X_t$ . Then, according to (Chen et al., 2008, Proposition 2.1 and Theorems 2.3 and 3.6), there exist positive constants  $c_i > 0$  ( $i = 1, \dots, 6$ ) such that for any  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$  with  $|x - y| \leq 1$ ,

$$(2.7) \quad c_2 \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \leq p_t(x, y) \leq c_1 \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right),$$

and for any  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$  with  $|x - y| > 1$ ,

$$(2.8) \quad c_5 \left( \frac{t}{|x - y|} \right)^{c_6|x-y|} \leq p_t(x, y) \leq c_3 \left( \frac{t}{|x - y|} \right)^{c_4|x-y|}.$$

Moreover, by (Chen et al., 2008, Proposition 2.2), there also exists a constant  $c_7 > 0$  such that for any  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$ ,

$$(2.9) \quad p_t(x, y) \leq c_7 t^{-d/\alpha}.$$

**Lemma 2.5.** *Let  $p_t(x, y)$  be the density function of the rotationally invariant symmetric truncated  $\alpha$ -stable Lévy process  $X_t$  above. Then, there are two positive constants  $C_1$  and  $C_2$  such that for any  $0 < t \leq 1$ ,  $x, y$  and  $z \in \mathbb{R}^d$ ,*

$$(2.10) \quad \frac{p_t(x, z)}{p_t(y, z)} \leq C_1 t^{-d/\alpha} \left( \frac{2 \vee |x - y| \vee |y - z|}{t} \right)^{C_2(2 \vee |x - y| \vee |y - z|)}.$$

*Proof.* Fix  $t \in (0, 1]$ . When  $|y - z| \leq t^{1/\alpha}$ , by (2.7),

$$p_t(y, z) \geq c_2 t^{-d/\alpha}.$$

Thus, according to (2.9),

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_7 t^{-d/\alpha}}{c_2 t^{-d/\alpha}} = \frac{c_7}{c_2}.$$

Similarly, when  $t^{1/\alpha} < |y - z| \leq 1$ ,

$$p_t(y, z) \geq \frac{c_2 t}{|x - y|^{d+\alpha}},$$

which implies that

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_7 t^{-d/\alpha}}{c_2 t / |y - x|^{-(d+\alpha)}} = \frac{c_7}{c_2 t^{d/\alpha+1}}.$$

Furthermore, by (2.8) and (2.9), if  $|y - z| \geq 1$ , then

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_7 t^{-d/\alpha}}{c_5 (t/|y - z|)^{c_6|y-z|}} \leq \frac{c_7}{c_5 t^{d/\alpha}} \left( \frac{|y - z|}{t} \right)^{c_6|y-z|}.$$

In particular, when  $1 \leq |y - z| < 2 \vee |x - y|$ ,

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \frac{c_7}{c_5 t^{d/\alpha}} \left( \frac{|y - z|}{t} \right)^{c_6|y-z|} \leq \frac{c_7}{c_5 t^{d/\alpha}} \left( \frac{2 \vee |x - y|}{t} \right)^{c_6(2 \vee |x - y|)}.$$

Combining with all the estimates above, for any  $0 < t \leq 1$ ,  $x, y$  and  $z \in \mathbb{R}^d$ ,

$$\frac{p_t(x, z)}{p_t(y, z)} \leq \begin{cases} C_1 t^{-d/\alpha} \left( \frac{2 \vee |x - y|}{t} \right)^{C_2(2 \vee |x - y|)}, & |y - z| \leq 2 \vee |x - y|, \\ C_3 t^{-d/\alpha} \left( \frac{|y - z|}{t} \right)^{C_4|y-z|}, & |y - z| > 2 \vee |x - y|. \end{cases}$$

It follows our desired assertion.  $\square$

**Proposition 2.6.** *Let  $P_t$  be the semigroup of the rotationally invariant symmetric truncated  $\alpha$ -stable Lévy process  $X_t$  above. Then for any  $t > 0$ ,  $x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$ ,*

$$(2.11) \quad P_t(\log f)(x) \leq \log P_t f(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right).$$

*Proof.* (1) For any  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$  and  $t > 0$ ,

$$(2.12) \quad \begin{aligned} P_t(\log f)(x) &= \int (\log f(z)) p_t(x, z) dz \\ &= \int (\log f(z)) \frac{p_t(x, z)}{p_t(y, z)} p_t(y, z) dz \\ &\leq \int \log \left( \frac{p_t(x, z)}{p_t(y, z)} \right) p_t(x, z) dz + \log \left( \int f(z) p_t(y, z) dz \right) \\ &= \int \log \left( \frac{p_t(x, z)}{p_t(y, z)} \right) p_t(x, z) dz + \log P_t f(y), \end{aligned}$$

where in the above inequality we have used the following Young inequality: for any probability measure  $\mu$ , if  $g, h \geq 0$  with  $\mu(g) = 1$ , then

$$\mu(gh) \leq \mu(g \log g) + \log \mu(e^h).$$



(2) According to (2.10), for any  $0 < t \leq 1$  and  $x, y, z \in \mathbb{R}^d$ ,

$$\log \frac{p_t(x, z)}{p_t(y, z)} \leq \log C_1 + \frac{d}{\alpha} \log \frac{1}{t} + C_2(2 \vee |x - y| \vee |y - z|) \log \left( \frac{2 \vee |x - y| \vee |y - z|}{t} \right).$$

On the other hand, if  $|y - z| \geq 2 \vee |x - y|$ , then

$$|x - z| \geq |y - z| - |x - y| \geq |y - z|/2 \geq 1,$$

and so, by (2.8),

$$p_t(x, z) \leq c_3 \left( \frac{t}{|x - z|} \right)^{c_4|x-z|} \leq c_3 \left( \frac{2t}{|y - z|} \right)^{c_4|y-z|/2}.$$

Therefore,

$$\begin{aligned} & \int \log \left( \frac{p_t(x, z)}{p_t(y, z)} \right) p_t(x, z) dz \\ & \leq \log C_1 + \frac{d}{\alpha} \log \frac{1}{t} \\ & \quad + C_2(2 \vee |x - y|) \log \left( \frac{2 \vee |x - y|}{t} \right) \int_{|y-z| \leq 2 \vee |x-y|} p_t(x, z) dz \\ & \quad + C_2 \int_{|y-z| > 2 \vee |x-y|} |y - z| \log \left( \frac{|y - z|}{t} \right) p_t(x, z) dz \\ (2.13) \quad & \leq \log C_1 + \frac{d}{\alpha} \log \frac{1}{t} + C_2(2 \vee |x - y|) \log \left( \frac{2 \vee |x - y|}{t} \right) \\ & \quad + C_3 \int_{|y-z| > 2 \vee |x-y|} |y - z| \left( \frac{2t}{|y - z|} \right)^{c_4|y-z|/2} \log \left( \frac{|y - z|}{t} \right) dz \\ & \leq \log C_1 + \frac{d}{\alpha} \log \frac{1}{t} + C_2(2 \vee |x - y|) \log \left( \frac{2 \vee |x - y|}{t} \right) \\ & \quad + C_4 t^{c_4} (1 + \log t^{-1}) \int_{|y-z| > 2} |y - z|^{-c_4|y-z|/2+1} \log |y - z| dz \\ & \leq C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t} \right). \end{aligned}$$

(3) For  $t \in (0, 1]$ , the required assertion follows from (2.12) and (2.13). For any  $t > 0$  and  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$ ,

$$\begin{aligned} P_t(\log f)(x) &= P_{t \wedge 1} P_{(t-1)^+}(\log f)(x) \\ &\leq P_{t \wedge 1}(\log P_{(t-1)^+} f)(x) \\ &\leq \log (P_{t \wedge 1} P_{(t-1)^+} f)(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right) \\ &= \log P_t f(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right), \end{aligned}$$

where the first inequality follows from the Jensen inequality: for any probability measure  $\mu$  and  $f \geq 0$ ,  $\mu(\log f) \leq \log \mu(f)$ , and in the second inequality we have used the fact that  $P_{(t-1)^+}f \geq 1$  and the conclusion for  $t \in (0, 1]$  proved in the previous two steps. We have proved our desired assertion.  $\square$

**Remark 2.7.** By the proof of Proposition 2.6, the inequality (2.11) is based on estimates (2.7), (2.8) and (2.9). Therefore, (2.11) also holds for Lévy type processes, whose Lévy jump kernel is given by  $j(x, y) = \frac{c(x, y)}{|x - y|^{d+\alpha}} \mathbf{1}_{\{|x - y| \leq r\}}$ . Here,  $r \in (0, \infty)$  and  $c(x, y)$  is a symmetric function on  $\mathbb{R}^d \times \mathbb{R}^d$  that is bounded upper and below by two positive constants, see e.g. (Chen et al., 2008).

We finally turn to the proof of Theorem 1.4.

*Proof of Theorem 1.4.* Under the condition in Theorem 1.4, we can write

$$X_t = X'_t + X_t^T,$$

where  $X_t^T$  is a rotationally invariant symmetric truncated  $\alpha$ -stable Lévy process with Lévy measure

$$\nu_{X^T}(dz) := c|z|^{-d-\alpha} \mathbf{1}_{\{|z| \leq r\}} dz,$$

and  $X'_t$  is a Lévy process with Lévy measure

$$\nu_{X'}(dz) := \nu(dz) - c|z|^{-d-\alpha} \mathbf{1}_{\{|z| \leq r\}} dz \geq 0.$$

Let  $P_t$ ,  $P_t^T$  and  $P'_t$  be the semigroups of the Lévy processes  $X_t$ ,  $X_t^T$  and  $X'_t$ , respectively. Then, according to Proposition 2.6, there exists  $C > 0$  such that for any  $t > 0$ ,  $x, y \in \mathbb{R}^d$  and  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$ ,

$$P_t^T(\log f)(x) \leq \log P_t^T f(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right).$$

Therefore, for  $t > 0$  and  $f \in B_b^+(\mathbb{R}^d)$  with  $f \geq 1$ ,

$$\begin{aligned} P_t(\log f)(x) &= P_t^T P'_t(\log f)(x) \\ &\leq P_t^T \log(P'_t f)(x) \\ &\leq C \log(P_t^T P'_t f)(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right) \\ &= C \log P_t f(y) + C(1 + |x - y|) \log \left( \frac{2 + |x - y|}{t \wedge 1} \right), \end{aligned}$$

where in the above two inequalities we have used the Jensen inequality and the fact that  $P'_t f \geq 1$ , respectively. The proof is complete.  $\square$

**Acknowledgement.** The author would like to thank Professor Feng-Yu Wang for comments on earlier versions of the paper, and to thank Professor René L. Schilling for helpful conversations. Financial support through the Alexander-von-Humboldt Foundation and the Natural Science Foundation of Fujian (No. 2010J05002) is also gratefully acknowledged. He also would like to thank an anonymous referee for carefully corrections on the first draft of this paper.

## REFERENCES

- Arnaudon, M., Thalmaier, A. and Wang, F.Y., 2006. Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below, *Bull. Sci. Math.* **130**, 223–233.
- Blumenthal, R.M. and Gettoor, R.K., 1960. Some theorems on stable processes. *Trans. Amer. Math. Soc.* **95**, 263–273.
- Bobkov, S.G., Gentil, I. and Ledoux, M., 2001. Hypercontractivity of Hamilton-Jacobi equations, *J. Math. Pures Appl.* **80**, 669–696.
- Bogdan, K. and Jakubowski, T., 2007. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, *Commun. Math. Phys.* **271**, 179–198.
- Bogdan, K., Stós, A. and Sztonyk, P., 2003. Harnack inequality for stable processes on  $d$ -sets, *Studia Math.* **158**, 163–198.
- Chen, Z.Q. and Kumagai, T., 2003. Heat kernel estimates for stable-like processes on  $d$ -sets, *Stochastic Process Appl.* **108**, 27–62.
- Chen, Z.Q. and Kumagai, T., 2008. Heat kernel estimates for jump processes of mixed types on metric measure spaces, *Probab. Theory Relat. Fields* **140**, 277–317.
- Chen, Z.Q., Kim, P. and Kumagai, T., 2008. Weighted Poincaré inequality and heat kernel estimates for finite range jump processes, *Math. Ann.* **342**, 833–883.
- Gordina, M., Röckner, M. and Wang, F.Y., 2011. Dimension-independent Harnack inequalities for subordinated semigroups, *Potential Anal.* **34**, 293–307.
- Jacob, N., 2001. *Pseudo Differential Operators and Markov Processes. Vol. 1: Fourier Analysis and Semigroups*, Imperial College Press, London.
- Ouyang, S.X., 2009. *Harnack inequalities and applications for stochastic equations*, Ph.D. thesis, Bielefeld University, 2009.
- Ouyang, S.X., Röckner, M. and Wang, F.Y., 2011. Harnack inequalities and applications for Ornstein-Uhlenbeck semigroups with jumps, *to appear in Potential Anal.*, see also arXiv:0908.2889
- Röckner, M. and Wang, F.Y., 2003. Harnack and functional inequalities for generalized Mehler semigroups, *J. Funct. Anal.* **203**, 237–261.
- Wang, F.Y., 1997. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, *Probab. Theory Related Fields* **109**, 417–424.
- Wang, F.Y., 2006a. A Harnack inequality for non-symmetric Markov semigroups, *J. Funct. Anal.* **239**, 297–309.
- Wang, F.Y., 2006b. Dimension-free Harnack inequality and its applications, *Front. Math. China* **1**, 53–72.
- Wang, F.Y., 2007. Harnack inequality and applications for stochastic generalized porous media equations, *Ann. Probab.* **35**, 1333–1350.
- Wang, F.Y., 2010a. Harnack inequalities on manifolds with boundary and applications, *J. Math. Pures Appl.* **94**, 304–321.
- Wang, F.Y., 2010b. Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on non-convex manifolds, *to appear in Ann. Probab.*, see also arXiv:0911.1644v3
- Wang, F.Y., 2011. Gradient estimate for Ornstein-Uhlenbeck jump processes, *Stoch. Proc. Appl.* **121**, 466–478.